A Factorization Theorem for Banded Matrices

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ABSTRACT

In this paper we prove a factorization theorem for strictly m-banded totally positive matrices. We show that such a matrix is a product of m one-banded matrices with positive entries.

A bi-infinite matrix $A=(A_{i,j})$, $-\infty<i,j<\infty$, is called strictly m-banded if $A_{i,j}=0$ for $j<i-m$, $j>i$, and $A_{i,i-m}A_{i,i}\neq 0$. These matrices arise quite frequently in applications, and recently they have been the subject of several independent investigations [2,3].

In this paper we prove a factorization theorem for *strictly m-banded totally positive matrices*. We show that such a matrix is a product of m one-banded matrices with positive entries.

Let us begin by recalling that a matrix A is *totally positive* if and only if *all* its minors are nonnegative:

$$A\begin{pmatrix} i_1,\dots,i_s \\ j_1,\dots,j_s \end{pmatrix}=\begin{vmatrix} A_{i_1,j_1} & \cdots & A_{i_1,j_s} \\ \vdots & & \vdots \\ A_{i_s,j_1} & \cdots & A_{i_s,j_s} \end{vmatrix}\geqslant 0,$$

$i_1<\cdots<i_s$, $j_1<\cdots<j_s$.

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For any positive sequence $\{r_i\}$ we define R to be the one-banded matrix given by

$$R_{i,i}=1, \qquad R_{i,i-1}=r_{i-1}.$$

Any matrix of this type will be called *elementary.*
Our main theorem is

THEOREM 1. *Let A be a strictly m-banded totally positive matrix normalized so that $A_{i,i}=1$. Then*

$$A=R_1\cdots R_m,$$

where each R_j is elementary.

It is well known that the product of totally positive matrices is totally positive. Hence, Theorem 1 gives a complete characterization of strictly m-banded totally positive matrices. Specifically, they have the form $R_1\cdots R_m B$ where B is a diagonal matrix with positive entries and each R_j is elementary. We prove Theorem 1 in a series of ancillary results of independent interest.

A *lower triangular* matrix A has the characteristic property that $A_{i,j}=0$, $j>i$. A finite lower triangular matrix is invertible if and only if $A_{i,i}\neq 0$, and the same holds for a bi-infinite matrix when we restrict our attention to *lower triangular* inverses. To make this precise, we introduce for any indices $p\leqslant q$ the submatrices

$$A^{p,q}=(A_{i,j}), \qquad p\leqslant i,j\leqslant q.$$

Here we use the same indexing for the entries of A and $A^{p,q}$; thus $(A^{p,q})_{p,p}$ is the upper left-hand element in $A^{p,q}$. With this notation one observes that if A and B are lower triangular matrices, then

$$(AB)^{p,q}=A^{p,q}B^{p,q}.$$

Exploiting this simple observation, we have

LEMMA 1. *If A is a lower triangular bi-infinite matrix and $A_{i,i}\neq 0$, then A has a (unique) lower triangular inverse which is given by*

$$A^{-1}_{i,j}=(A^{p,q})^{-1}_{i,j}$$

for any p, q with $p \leqslant i \leqslant q$, $p \leqslant j \leqslant q$.

Proof. The observation shows that B is a lower triangular inverse of A if and only if $B^{p,q} = (A^{p,q})^{-1}$ for all $p \leqslant q$. In particular, B is then the unique lower triangular inverse of A, which we denote henceforth by A^{-1}. ■

We will use the following notation. The diagonal matrix with diagonal entries $D_{i,i} = (-1)^i$ will be denoted by D. Also, we set $|A| = (|A_{i,j}|)$.

PROPOSITION 1. *Let A be a lower triangular bi-infinite matrix which is totally positive, and $A_{i,i} > 0$. Suppose A^{-1} is the lower triangular inverse of A. Then $DA^{-1}D = |A^{-1}|$, and $|A^{-1}|$ is totally positive.*

Proof. We wish to show that

$$|A^{-1}|\begin{pmatrix} i_j, \dots, i_s \\ j_1, \dots, j_s \end{pmatrix} \geqslant 0$$

for $i_1 < \cdots < i_s, j_1 < \cdots < j_s$. We do this by choosing p, q with $p \leqslant i_1 < \cdots < i_p \leqslant q, p \leqslant j_1 < \cdots < j_s \leqslant q$ so that

$$|A^{-1}|\begin{pmatrix} i_1, \dots, i_s \\ j_1, \dots, j_s \end{pmatrix} = |(A^{p,q})^{-1}|\begin{pmatrix} i_1, \dots, i_s \\ j_1, \dots, j_s \end{pmatrix}.$$

It is known that for finite matrices $D(A^{p,q})^{-1}D = |(A^{p,q})^{-1}|$ is totally positive. Thus the theorem is proved. ■

We will eventually apply the proposition to the lower triangular inverse of a strictly m-banded matrix. But first we prove

THEOREM 2. *Let A be a strictly m-banded totally positive matrix. If i_l, j_l are indices satisfying $0 \leqslant i_l - j_l \leqslant m$, $l = 1, \dots, s$, then*

$$A\begin{pmatrix} i_1, \dots, i_s \\ j_1, \dots, j_s \end{pmatrix} > 0, \qquad 0 \leqslant i_l - j_l \leqslant m, \quad l = 1, \dots, s.$$

Proof. We will prove this result by induction on s. First we verify that $A_{i,j} > 0$, $0 \leqslant i - j \leqslant m$. If $i - j = 0$ or $i - j = m$, the strict bandedness of A

insures the positivity of $A_{i,j}$. For $0<i-j<m$, we have

$$0\leqslant A\begin{pmatrix} j & i \\ i-m & j \end{pmatrix} = \begin{vmatrix} A_{j,i-m} & A_{j,j} \\ A_{i,i-m} & A_{i,j} \end{vmatrix},$$

which gives $A_{j,i-m}A_{i,j}\geqslant A_{j,j}A_{i,i-m}>0$, insuring that $A_{i,j}>0$. Now, suppose the theorem is true for all $s\times s$ minors. We wish to show that

$$\Delta = A\begin{pmatrix} i_1, i_2, \ldots, i_{s+1} \\ j_1, j_2, \ldots, j_{s+1} \end{pmatrix} > 0$$

for $0\leqslant i_l - j_l \leqslant m$, $l=1,\ldots,s+1$.

If $i_1-j_1=0$ then $A_{i_1,j_l}=0$, $l=2,\ldots,s+1$, and so

$$\Delta = A_{i_1,j_1}A\begin{pmatrix} i_2, \ldots, i_{s+1} \\ j_2, \ldots, j_{s+1} \end{pmatrix} > 0$$

by the induction hypothesis. Similarly, if $i_1-j_1=m$, then $A_{i_l j_1}=0$, $l=2,\ldots,s+1$, and again we have

$$\Delta = A_{i_1,j_1}A\begin{pmatrix} i_2, \ldots, i_{s+1} \\ j_2, \ldots, j_{s+1} \end{pmatrix} > 0.$$

In the case that $0<i_1-j_1<m$, we use Sylvester's determinant identity which gives

$$\begin{aligned} 0 &\leqslant A\begin{pmatrix} j_1, i_1, i_2, \ldots, i_{s+1} \\ i_1-m, j_1, j_2, \ldots, j_{s+1} \end{pmatrix} A\begin{pmatrix} i_2, \ldots, i_{s+1} \\ j_2, \ldots, j_{s+1} \end{pmatrix} \\ &= \begin{vmatrix} A\begin{pmatrix} j_1, i_2, \ldots, i_{s+1} \\ i_1-m, j_2, \ldots, j_{s+1} \end{pmatrix} & A\begin{pmatrix} j_1, i_2, \ldots, i_{s+1} \\ j_1, j_2, \ldots, j_{s+1} \end{pmatrix} \\ A\begin{pmatrix} i_1, i_2, \ldots, i_{s+1} \\ i_1-m, j_2, \ldots, j_{s+1} \end{pmatrix} & A\begin{pmatrix} i_1, \ldots, i_{s+1} \\ j_1, \ldots, j_{s+1} \end{pmatrix} \end{vmatrix} \end{aligned}$$

Since we have already shown that the minors in the lower left and upper right positions in this matrix are positive, it follows that

$$A\begin{pmatrix} i_1,\ldots,i_{s+1} \\ j_1,\ldots,j_{s+1} \end{pmatrix} > 0$$

as claimed. ∎

We now can combine Proposition 1 and Theorem 2 to obtain

THEOREM 3. *If A is a strictly m-banded, totally positive matrix and A^{-1} is its lower triangular inverse, then as long as $m \geqslant 1$,*

$$(-1)^{i+j} A^{-1}_{i,j} > 0, \qquad j \leqslant i.$$

Proof. Pick p, q so that $p \leqslant i, j \leqslant q$; then using Lemma 1 and Cramer's rule,

$$(-1)^{i+j} A^{-1}_{i,j} = (-1)^{i+j} (A^{p,q})^{-1}_{i,j}$$

$$= \frac{A\begin{pmatrix} p,\ldots,j-1,j+1,\ldots,q \\ p,\ldots,i-1,i+1,\ldots,q \end{pmatrix}}{A\begin{pmatrix} p,\ldots,q \\ p,\ldots,q \end{pmatrix}}.$$

According to Theorem 2, these minors are all positive, which proves the theorem. ∎

LEMMA 2. *Let A be a bi-infinite, totally positive lower triangular matrix with $A_{i,j} > 0$, $i > j$. Then for each j, $A_{i,j}/A_{i,j+1}$ is a nonincreasing function of i $(i > j+1)$ which converges to*

$$r_j(A) = \lim_{i \to \infty} \frac{A_{i,j}}{A_{i,j+1}}.$$

Proof. The inequality

$$0 \leqslant \begin{vmatrix} A_{i,j} & A_{i,j+1} \\ A_{i+1,j} & A_{i+1,j+1} \end{vmatrix}$$

gives

$$\frac{A_{i,j}}{A_{i,j+1}} \geqslant \frac{A_{i+1,j}}{A_{i+1,j+1}},$$

from which the lemma readily follows. ∎

THEOREM 4. *If A is a strictly m-banded totally positive matrix, then for all j*

$$r_j(|A^{-1}|) > 0.$$

Proof. Fix j and consider all $i > j$. Then

$$0 = \sum_{j}^{j+m} A_{i,l}^{-1} A_{l,j},$$

which gives

$$(-1)^j |A^{-1}|_{i,j} A_{j,j} + (-1)^{j+1} |A^{-1}|_{i,j+1} A_{j+1,j} + \cdots$$
$$+ (-1)^{j+m-1} |A^{-1}|_{i,j+m-1} A_{j+m-1,j} + (-1)^{j+m} |A^{-1}|_{i,j+m} A_{j+m,j} = 0.$$

Dividing both sides of this equation by $|A^{-1}|_{i,j+m}$ and letting $i \to \infty$ gives

$$r_j(A) \cdots r_{j+m-1}(A)(-1)^j A_{j,j} + r_{j+1}(A) \cdots r_{j+m-1}(A)(-1)^{j+1} A_{j+1,j}$$
$$+ \cdots + r_{j+m-1}(A)(-1)^{j+m-1} A_{j+m-1,j} + (-1)^{j+m} A_{j+m,j} = 0.$$

Now if $r_s(A) = 0$, we choose $j = s - m + 1$ above. This gives $A_{j+m,j} = 0$, which is a contradiction. ∎

Let $S = S(A)$ be the one-banded matrix defined by $S_{i,i} = 1$, $S_{i,i-1} = -r_{i-1}(A)$.

THEOREM 5. *Let A be a lower triangular totally positive matrix, $A_{i,j}>0$, $j<i$. Then $T=AS$ is also a lower triangular totally positive matrix.*

Proof. Pick any integers N, p, q, i with $p+N\leqslant i$ and $N\geqslant 0$. Then the matrix

$$\begin{pmatrix} A_{p,q} & \cdots & A_{p,q+N} \\ A_{p+1,q} & \cdots & A_{p+1,q+N} \\ \vdots & & \vdots \\ A_{p+N,q} & \cdots & A_{p+N,q+N} \\ A_{i,q} & \cdots & A_{i,q+N} \end{pmatrix}$$

is totally positive. Using an idea which appears to have originated in [1], we divide the last row by $A_{i,q+N}$, let $i\to\infty$, and conclude that the matrix

$$\begin{pmatrix} A_{p,q} & A_{p,q+1} & \cdots & A_{p,q+N-1} & A_{p,q+N} \\ \vdots & \vdots & & \vdots & \vdots \\ A_{p+N,q} & A_{p+N,q+1} & \cdots & A_{p+N,q+N-1} & A_{p+N,q+N} \\ r_q\cdots r_{q+N-1} & r_{q+1}\cdots r_{q+N-1} & \cdots & r_{q+N-1} & 1 \end{pmatrix}$$

is also totally positive.

Since

$$T_{i,j}=\sum A_{i,l}S_{l,j}=A_{i,j}-r_jA_{i,j+1},$$

Gaussian elimination on the last row of the above matrix gives

$$\begin{pmatrix} T_{p,q} & T_{p,q+1} & \cdots & T_{p,q+N-1} & A_{p,q+N} \\ \vdots & \vdots & \vdots & \vdots & \\ T_{p+N,q} & T_{p+N,q+1} & \cdots & T_{p+N,q+N-1} & A_{p+N,q+N} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix},$$

and thus by a result of A. Whitney [4] the matrix $(T_{i,j})$ $p\leqslant i\leqslant p+N$, $q\leqslant j\leqslant q+N-1$, is totally positive. Since p, q, N were arbitrarily chosen, the proof is complete. ∎

We are now ready to prove Theorem 1.

Proof. We apply Theorem 5 to $|A^{-1}|$ and conclude that $T=|A^{-1}|S$, $S=S(|A^{-1}|)$, is totally positive. According to Proposition 1, $|T^{-1}|=DT^{-1}D$ is totally positive. Moreover

$$|S||T^{-1}|=(DSD)\Big(D\big(|A^{-1}|S\big)^{-1}D\Big)$$

$$=DSS^{-1}|A^{-1}|^{-1}D=|A^{-1}|^{-1}D=A.$$

Thus setting $B=|T^{-1}|$, $R=|S|$, we get

$$A=RB,$$

where (according to Proposition 1 and Theorem 4) R is elementary and B is totally positive. Since R is obviously a "band expander," B must be strictly $(m-1)$-banded with $B_{i,i}=1$.

We may now continue to apply this factorization procedure to the residual matrix B until we finally factor A as a product of m elementary matrices. This proves Theorem 1. ■

We can modify the procedure given in Theorem 5 and obtain another factorization of banded matrices. Working with ratios of adjacent rows (rather than columns as in our previous work) we obtain the values

$$c_i(A)=\lim_{j\to-\infty}\frac{A_{i+1,j}}{A_{i,j}}.$$

Using these values in a manner completely analogous to what has gone before, we obtain

$$A=BC_1,$$

where C_1 is an elementary matrix with $C_{i,i-1}=c_{i-1}(|A^{-1}|)$, and B is some $(m-1)$-banded totally positive matrix. If we repeat the process, we obtain finally a factorization

$$A=C_mC_{m-1}\cdots C_1$$

where each C_j is an elementary factor. We now specialize to the case when A is N-periodic.

Observe that if the matrix $A=(A_{i,j})$ is *N-periodic*, $A_{i+N,j+N}=A_{i,j}$, then so are its elementary factors as determined by Theorem 1. Hence, we have in this case

$$A=R_1\cdots R_m,$$

where each R_j is an elementary N-periodic matrix. In the study of the invertibility of N-periodic matrices, the symbol $S(z)$ defined by

$$(S(z))_{i,j}=\sum_{l=-\infty}^{\infty} A_{i,j+lN}z^{-l}, \qquad z\in\mathbb{C}, \quad 1\leqslant i,j\leqslant N,$$

is fundamental. Notice that $S(z)$ is a $N\times N$ matrix-valued polynomial when A is banded.

It is easily seen that if $A=BC$ then

$$S_A=S_BS_C.$$

Thus for N-periodic strictly m-banded, totally positive matrices, we have a complete factorization of its symbol

$$S(z)=\prod_{j=1}^{m}\begin{pmatrix} 1 & 0 & \cdots & 0 & r_{1,j}z \\ r_{2,j} & 1 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & 1 & 0 \\ 0 & 0 & \cdots & r_{N,j} & 1 \end{pmatrix},$$

where $r_{i,j}>0$, $i=1,\dots,N$, $j=1,\dots,m$.

Since the determinant of the individual factors above are $1+(-1)^{N+1}\gamma_j z$, $\gamma_j=r_{1,j}\cdots r_{N,j}$, we obtain

$$\det S(z)=\prod_{j=1}^{m}\left[1+(-1)^{N+1}\gamma_j z\right].$$

This equation shows that $\det S(z)$ is a polynomial of *exact* degree m with real zeros having a common sign $(-1)^N$. We proved this fact earlier by other means in [3].

We note further that the factorization process for $S(z)$ given above yields a special order in the factors. Indeed, we can prove

THEOREM 6. $\gamma_j > \gamma_{j+1}$, $j=1,\dots,m-1$.

Thus in the factorization of $S(z)$ given above, the first factor corresponds to the zero of det $S(z)$ which has the smallest modulus, and in general the jth factor provides the jth zero of det $S(z)$, provided we have ordered the zeros of det $S(z)$ according to increasing modulus.

As our factorization is an iterative process, for Theorem 6 it is sufficient to prove the following lemma, which is then applied to $|A^{-1}|$ just as in the proof of Theorem 1.

LEMMA 3. *Let A be an N-periodic matrix which satisfies the hypotheses of Theorem 5. Then $r_j(A)\cdots r_{j+N-1}(A) \geqslant r_j(T)\cdots r_{j+N-1}(T)$, where T is as given in Theorem 5.*

Proof. From the N-periodicity of A we obtain, independently of j,

$$\mu = r_j \cdots r_{j+N-1},$$

where as before in Lemma 2

$$r_j = r_j(A) = \lim_{i\to\infty} \frac{A_{i,j}}{A_{i,j+1}}.$$

So

$$\mu = \lim_{i\to\infty} \frac{A_{i,j}}{A_{i,j+N}}, \qquad j \text{ any integer}.$$

Similarly, referring back to the proof of Theorem 5, we find

$$\mu' = r_j' \cdots r_{j+N-1}', \qquad j \text{ any integer},$$

where

$$r_j' = \lim_{i\to\infty} \frac{T_{i,j}}{T_{i,j+1}}$$

and

$$T_{i,j} = A_{i,j} - r_j A_{i,j+1}.$$

So

$$\mu' = \lim_{i\to\infty} \frac{T_{i,j}}{T_{i,j+N}} = \lim_{i\to\infty} \frac{A_{i,j} - r_j A_{i,j+1}}{A_{i,j+N} - r_{j+N} A_{i,j+N+1}} = \lim_{i\to\infty} e_i,$$

where, since $r_j = r_{j+N}$, we can set

$$e_i = \frac{A_{i,j} - r_j A_{i,j+1}}{A_{i,j+N} - r_j A_{i,j+N+1}}, \qquad j \text{ fixed.}$$

Observe

$$e_i - \frac{A_{i,j+1}}{A_{i,j+N+1}} = \frac{A_{i,j}A_{i,j+N+1} - A_{i,j+1}A_{i,j+N}}{\left(A_{i,j+N} - r_j A_{i,j+N+1}\right)A_{i,j+N+1}},$$

and this difference is nonpositive, since the numerator on the right-hand side is the negative of the determinant

$$\begin{vmatrix} A_{i-N,j} & A_{i-N,j+1} \\ A_{i,j} & A_{i,j+1} \end{vmatrix}.$$

Thus

$$e_i \leqslant \frac{A_{i,j+1}}{A_{i,j+N-1}}.$$

Passing i to infinity, we find $\mu' \leqslant \mu$ as desired. ■

Let us now add a few comments on the uniqueness of factorizations of the type given in Theorem 1. In general, such factorizations are not unique. For example, consider the two-banded Toeplitz matrix A determined by the sequence

$$\ldots, 0, 0, a, b, 1, 0, 0, \ldots, \qquad b^2 - 4a > 0, \quad a, b > 0.$$

Let R_1 and R_2 be elementary factors with $(R_1)_{i,i-1} = \rho_{i-1}$, $(R_2)_{i,i-1} = \delta_{i-1}$. Then $A = R_1 R_2$ is equivalent to

$$a = \rho_i \delta_{i-1},$$

$$b = \rho_i + \delta_i$$

for all i, and hence

$$\delta_i = b - \frac{a}{\delta_{i-1}}.$$

We have to choose appropriate starting values δ_0 so that the above iteration yields positive δ_i for all i. Because of the total positivity, the quadratic ax^2+bx+1 has only negative zeros $-x_1 < -x_0 < 0$. Using this fact, one easily verifies that the above iteration yields $\delta_i > 0$ for all i provided $x_0 \leqslant \delta_0 \leqslant x_1$. If $\delta_0 = x_0$ or $\delta_0 = x_1$, the two elementary factors R_1 and R_2 are Toeplitz, and this yields the only factorization into elementary Toeplitz factors. But if $x_0 < \delta_0 < x_1$, we obtain non-Toeplitz elementary factors.

Recalling that the factorization given in Theorem 1 produces, for an N-periodic matrix, elementary factors which are themselves N-periodic, we might expect such a periodic factorization to be unique. Up to a prescribed ordering of the zeros of det $S(z)$ this conjecture is correct. To understand the situation fully, let us consider the following simple two-periodic, two-banded example:

$$A = \begin{pmatrix} \cdots & & & & & & & \\ \cdots & 0 & a & c & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & d & b & 1 & 0 & \cdots \\ & & & & & & & \cdots \end{pmatrix}.$$

By the remarks following Theorem 1, we know that $S_A(z)$ factors as

$$\begin{pmatrix} 1+az & cz \\ b & 1+dz \end{pmatrix} = \begin{pmatrix} 1 & \gamma_1\alpha^{-1}z \\ \alpha & 1 \end{pmatrix} \begin{pmatrix} 1 & \gamma_2\beta^{-1}z \\ \beta & 1 \end{pmatrix},$$

where $\alpha, \beta, \gamma_1, \gamma_2$ are all positive and γ_1^{-1} and γ_2^{-1} are the two zeros of $S_A(z)$. The fact that $\gamma_1 \geqslant \gamma_2$ (see Proposition 1) distinguishes this factorization. We can find another factorization of the form

$$\begin{pmatrix} 1 & \gamma_2\tilde{\alpha}^{-1}z \\ \tilde{\alpha} & 1 \end{pmatrix} \begin{pmatrix} 1 & \gamma_1\tilde{\beta}^{-1}z \\ \tilde{\beta} & 1 \end{pmatrix} = \begin{pmatrix} 1+\gamma_2\tilde{\alpha}^{-1}\tilde{\beta}z & z(\tilde{\alpha}^{-1}\gamma_2 + \tilde{\beta}^{-1}\gamma_1) \\ \tilde{\alpha}+\tilde{\beta} & 1+\tilde{\alpha}\tilde{\beta}^{-1}\gamma_1 z \end{pmatrix}.$$

For such a factorization of $S_A(z)$ we require

$$\tilde{\alpha}^{-1}\tilde{\beta}\gamma_2=a,$$

$$\tilde{\alpha}+\tilde{\beta}=b,$$

$$\tilde{\alpha}^{-1}\gamma_0+\tilde{\beta}^{-1}\gamma_1=c,$$

$$\tilde{\alpha}\tilde{\beta}^{-1}\gamma_1=d.$$

Clearly, the first two equations can be solved for $\tilde{\alpha}, \tilde{\beta}$. Then since $a=\alpha^{-1}\beta\gamma_1$, we get $d=\alpha\beta^{-1}\gamma_2=\tilde{\alpha}\tilde{\beta}^{-1}\gamma_1$, so that the last equation is automatically satisfied. To check the third equation, merely observe that

$$c-\frac{\gamma_2}{\tilde{\alpha}}-\frac{\gamma_1}{\tilde{\beta}}=\gamma_1\left(\frac{\tilde{\beta}-\alpha}{\alpha\tilde{\beta}}\right)+\gamma_2\left(\frac{\tilde{\alpha}-\beta}{\beta\tilde{\alpha}}\right)$$

$$=\frac{a}{\beta\tilde{\beta}}(\tilde{\beta}+\tilde{\alpha}-\alpha-\beta)=0.$$

Thus we indeed have two different factorizations of $S_A(z)$, provided only that the zeros of $\det S_A(z)$ are distinct. The proof of Theorem 7 given below will reveal that this second factorization arises from the column procedure outlined after the proof of Theorem 1.

As the above example suggests, the order of the zeros of $\det S_A(z)$ plays an essential role in the factorization of $S_A(z)$. This observation we formulate as

THEOREM 7. *Let $S(z)$ be the symbol of a strictly m-banded totally positive, N-periodic matrix A. If the zeros of* $\det S(z)$ *are listed as $z_1,\dots,z_m$, then to this ordering of the zeros there corresponds a unique factorization,*

$$S(z)=P_1(z)\cdots P_m(z)$$

where each $P_j(z)$ is the symbol of an N-periodic elementary factor and $\det P_j(z)=0$ *only when $z=z_j$.*

Proof. Let us first prove uniqueness. Suppose $S(z)=P_1(z)\cdots P_m(z)=Q_1(z)\cdots Q_m(z)$, where

$$\det P_i(z_i)=\det Q_i(z_i)=0, \qquad i=1,\dots,m.$$

The assertion being trivial for $m=1$, we proceed by induction on m and assume we have uniqueness when there are only m-1 factors. Suppose

$$P_m(z)=\begin{pmatrix} 1 & 0 & \cdots & 0 & p_1 z \\ p_2 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & p_N & 1 \end{pmatrix}$$

and

$$Q_m(z)=\begin{pmatrix} 1 & 0 & \cdots & 0 & q_1 z \\ q_2 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & q_N & 1 \end{pmatrix}.$$

Then $x=\{x_j\}_{j=1}^N$, $x_j=(-1)^{N-j}p_1\cdots p_j$, and $y=\{y_j\}_{j=1}^N$, $y_j=(-1)^{N-j}q_1\cdots q_j$, satisfy

$$P_m(z_m)x=Q_m(z_m)y=0,$$

and hence

$$S(z_m)x=S(z_m)y=0.$$

But from [3, Corollary 1], we know that the null space of the symbol $S(z_m)$ must be one-dimensional and hence $x=cy$ for some constant c. However, since by hypothesis $x_N=y_N$, we conclude $c=1$. So $P_m(z)=Q_m(z)$ and consequently

$$P_1(z)\cdots P_{m-1}(z)=Q_1(z)\cdots Q_{m-1}(z),$$

so that the assertion follows by induction.

We now turn to existence. We have already established, as a consequence of Theorem 1, the factorization which corresponds to $z_j=\gamma_j^{-1}$, $j=1,\ldots,m$, so that the zeros are given in order of increasing modulus. Any other listing $z_1,\ldots,z_n$ of the zeros is a permutation of $\gamma_1^{-1},\ldots,\gamma_m^{-1}$ and as such is a product of transpositions. It follows that the factorization given in the theorem will be established provided we can settle the special case when $m=2$.

So let A be a fixed two-banded, N-periodic, totally positive matrix. Using Theorem 1 and the remarks immediately following its proof, we obtain two factorizations:

$$S_A(z)=R_1(z)R_2(z)=C_2(z)C_1(z).$$

We will show that $\det R_1(z)=\det C_1(z)$, whence it will follow that $\det R_2(z)=\det C_2(z)$. Thus $S_A(z)$ is factored in two distinct ways corresponding to the two possible orderings of its zeros.

Set

$$\det R_1(z)=1+(-1)^{N+1}\gamma z,$$

$$\det C_1(z)=1+(-1)^{N+1}\eta z.$$

As in Lemma 3, we observe that

$$\gamma=\lim_{i\to\infty}\frac{A_{i,0}}{A_{i,N}}=\lim_{i\to\infty}\frac{A_{iN,0}}{A_{iN,N}}.$$

Similarly,

$$\eta=\lim_{j\to-\infty}\frac{A_{N,j}}{A_{0,j}}=\lim_{j\to-\infty}\frac{A_{N,jN}}{A_{0,jN}}.$$

Since the matrix A is N-periodic the submatrix $\{A_{iN,jN}\}_{i,j=-\infty}^{\infty}$ is a Toeplitz matrix, and as such we have

$$A_{iN,jN}=a_{i-j},$$

where

$$a_i=A_{iN,0}.$$

Then

$$\eta = \lim_{j\to-\infty} \frac{a_{i-j}}{a_{-j}} = \lim_{j\to\infty} \frac{a_{1+j}}{a_j} = \gamma,$$

as was desired. This completes the proof of Theorem 7. ■

As a final remark, we mention that not every m-banded, N-periodic matrix is factorizable into elementary factors. The following example of a two-periodic matrix,

$$A = \begin{pmatrix} \cdots & & & & & & & \\ \cdots & 0 & 6 & 0 & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 5 & 0 & 1 & 0 & \cdots \\ & & & & & & & \cdots \end{pmatrix},$$

has the symbol

$$S_A(z) = \begin{pmatrix} 1+6z & 0 \\ 0 & 1+5z \end{pmatrix}.$$

If A were to factor as the product of two elementary matrices, we would have

$$S_A(z) = \begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix} \begin{pmatrix} e(z) & f(z) \\ g(z) & h(z) \end{pmatrix},$$

where each of the components is constant or linear. From the orthogonality conditions implicit in the zeros of $S_A(z)$, we see that the second factor must have the form

$$\begin{pmatrix} -\alpha d(z) & -\beta b(z) \\ \alpha c(z) & \beta a(z) \end{pmatrix}$$

for some pair α, β. Hence,

$$(1+6z) = \alpha\left[c(z)b(z) - d(z)a(z)\right],$$

whereas

$$(1+5z) = \beta\left[d(z)a(z) - c(z)b(z)\right] = -\beta\alpha^{-1}(1+6z),$$

which is impossible. So A does not factor. Thus we see that total positivity is a convenient hypothesis which is sufficient to guarantee a factorization, but that mere nonnegativity of the matrix entries will not guarantee that the matrix factors.

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